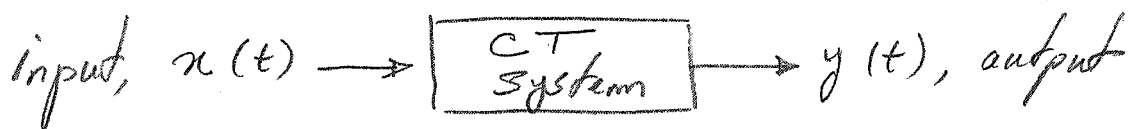


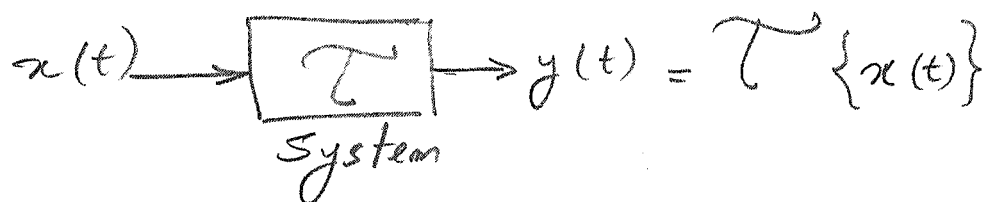
Continuous-time System (CT)

- * A system is a unit, a plant, a process that operates on an input and produces an output.
- * When the operation or behavior of a system is continuous function of time, the system is called Continuous-time system.
- * The input and output of a Continuous-time system are taken to be continuous-time signals.



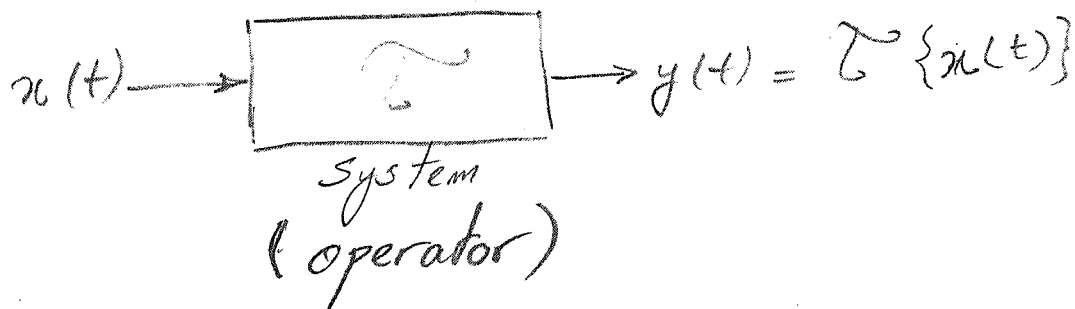
- * Many physical signals and systems are treated, i.e. modeled, analyzed, designed and tested, in time-domain, frequency domain, or both domains.

A simple reason is that some properties of the signal or system can be observed and used easier in one domain than in the other domain.



Classification of Continuous-time Systems

1. Linear and Non-linear Systems
2. Time-Varying and Time-Invariant Systems
3. Static (no memory) and Dynamic (with memory) systems
4. Causal and Non-causal systems
5. stable (Invertible) and unstable (Non-invertible) systems.



1. Linear and Non-linear Systems

$$x(t) \rightarrow \boxed{\mathcal{T}}_{\text{system}} \rightarrow y(t) = \mathcal{T}\{x(t)\}$$

$$x(t) \xrightarrow{\mathcal{T}} y(t)$$

* System $\mathcal{T}$ is Linear if the Superposition property holds.

Superposition property:

(1) Additivity property:

$$\text{If } x_k(t) \xrightarrow{\mathcal{T}} y_k(t) = \mathcal{T}\{x_k(t)\}$$

$$\text{for } k=1, 2, 3, \dots, M$$

then

$$x(t) = \sum_{k=1}^M x_k(t) \xrightarrow{\mathcal{T}} y(t) = \sum_{k=1}^M y_k(t)$$

(2) Multiplicative property:

$$\text{If } x(t) \xrightarrow{\mathcal{T}} y(t) = \mathcal{T}\{x(t)\}$$

then

$$v(t) = a x(t) \xrightarrow{\mathcal{T}} u(t) = a y(t)$$

for any constant a .

Thus Superposition Property can be written as

$$x(t) = \sum_{k=1}^M a_k x_k(t) \xrightarrow{\mathcal{T}} y(t) = \sum_{k=1}^M a_k y_k(t)$$

Note that $x_k(t)$ can be any input signal giving the output signal $y_k(t)$ for the system $\mathcal{T}$.

If the Superposition is not held then $\mathcal{T}$ is a Non-linear System.

Ex 1 $y(t) = \alpha x(t) + \beta$ α and β are constants.

That is $x(t) \xrightarrow{\mathcal{T}} y(t) = \alpha x(t) + \beta$

Testing for linearity - superposition property

For any $v(t) \xrightarrow{\mathcal{T}} u(t) = \alpha v(t) + \beta$

and $h(t) \xrightarrow{\mathcal{T}} p(t) = \alpha h(t) + \beta$

Additivity Test:

$$r(t) = v(t) + h(t) \xrightarrow{\mathcal{T}} g(t) = \alpha v(t) + \alpha h(t) + \beta$$

but

$$p(t) + u(t) = \alpha v(t) + \alpha h(t) + 2\beta$$

So $g(t) \neq p(t) + v(t) \Rightarrow$ system is not linear.

Multiplicative Test:

$$m(t) = c \cdot v(t) \xrightarrow{\mathcal{T}} z(t) = \alpha c v(t) + \beta \\ = \alpha m(t) + \beta$$

But $c \cdot u(t) = \alpha c v(t) + c\beta \neq z(t)$
 $\Rightarrow$ system is not linear.

However if $\beta = 0$ then system $y(t) = \alpha x(t)$ is linear since both additivity and multiplicative, thus superposition, properties will hold.

$\rightarrow$ See Also examples given in the Textbook.

2. Time-Invariant and Time-Varying Systems

$$x(t) \rightarrow \boxed{\mathcal{T}} \rightarrow y(t) = \mathcal{T}\{x(t)\}$$

* System $\mathcal{T}$ is Time-Invariant (TI) if the behavior of the system does not change over time, otherwise the system is Time-Varying.

That is if for any $x(t) \xrightarrow{\mathcal{T}} y(t) = \mathcal{T}\{x(t)\}$

we have

$$v(t) = x(t-\alpha) \xrightarrow{\mathcal{T}} z(t) = \mathcal{T}\{v(t)\} \\ = \mathcal{T}\{x(t-\alpha)\} \\ = y(t-\alpha)$$

then $\mathcal{T}$ is Time-Invariant,

otherwise it is Time-Varying.

Ex1

$$u(t) = t v(t)$$

Test:

$$(1) \quad u(t-\alpha) = (t-\alpha) v(t-\alpha)$$

$$(2) \quad \text{Let input be } g(t) = v(t-\alpha)$$

$$\text{Then the output } p(t, \alpha) = t v(t-\alpha)$$

Since $u(t-\alpha) \neq p(t, \alpha)$ then the system is time-Variant.

Ex2 The same as Example 2.2.4 (b):

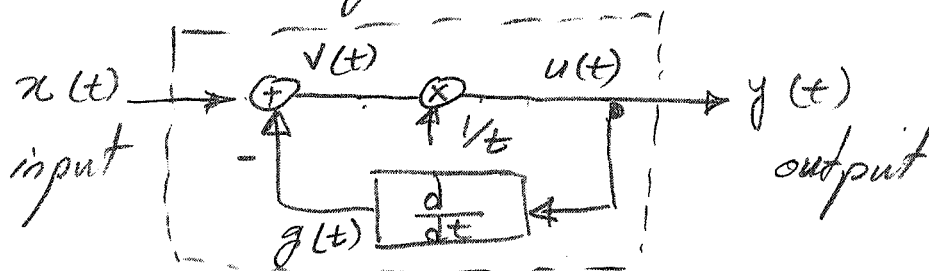
$$\frac{dy(t)}{dt} = -t y(t) + x(t) \quad t \geq 0, \quad y(0) = 0.$$

In this example $x(t)$ is input and $y(t)$ is the output.

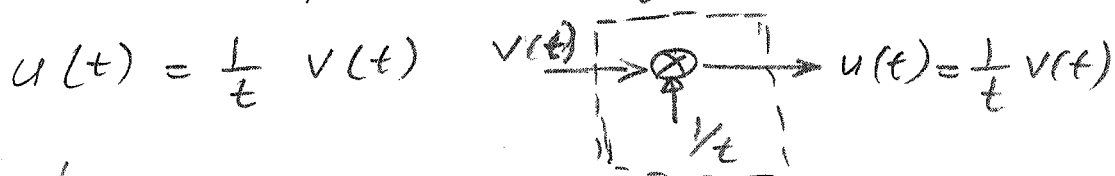
We can show the system as

$$y(t) = + \frac{1}{t} \left\{ - \frac{dy(t)}{dt} + x(t) \right\}$$

And draw the system as



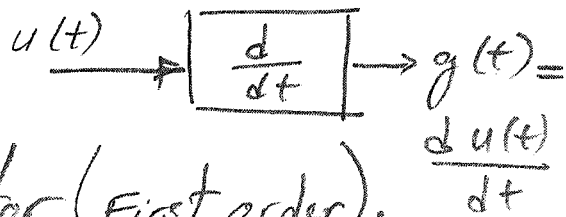
where in one part of the system we have



Just like Ex 1 in previous page we can show that the above part of the system is Time-Varying. Thus the original system is Time-Varying.

Ex 3 Note that the other part of the system in Ex 2 is

$$g(t) = \frac{d}{dt} u(t)$$



This is the Differentiator (First order).

Test for Time-Invariant or Time-Varying:

(1) Let $w(t) = u(t - \alpha)$ then from the equation:

$$g(t - \alpha) = \frac{d}{dt} u(t - \alpha)$$

(2) Now $w(t)$ as input yields

$$p(t, \alpha) = \frac{dw(t)}{dt} = \frac{d}{dt} u(t - \alpha)$$

But $g(t-\alpha) = p(t-\alpha)$

The the system, The differentiation, is
Time-Invariant.

For example: Let $u(t) = at^3 - bt + c$

then $u(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow g(t) = \frac{du(t)}{dt} = 3at^2 - b$

Test
(1) $g(t-\alpha) = \frac{d(u(t-\alpha))}{dt} = 3a(t-\alpha)^2 - b$

$$g(t-\alpha) = 3at^2 - 6at\alpha + 3a\alpha^2 - b$$

(2) $u(t-\alpha) = a(t-\alpha)^3 - b(t-\alpha) + c$

$$p(t, \alpha) = \frac{d u(t-\alpha)}{dt} = 3a(t-\alpha)^2 - b = g(t-\alpha)$$

Thus system is Time-Invariant.

3. Systems with and without Memory

(A) Memoryless system: The output at time t depends on the present, or time t , of the input

Ex

(1) $y(t) = 3t^5 - 2t^2 + 11$ ONLY.

or $y(t) = 3x^5(t) - 2x^2(t) + 11$ where $x(t) = t \geq 0$

(2) $y(t) = a x(t)$

(3) $y(t) = t^3 x(t)$

The above systems are Memoryless

(B) Systems with Memory: The output at time t depends on present and past values of the input.

Ex

(1) $\frac{dy(t)}{dt} = x(t)$

that is $y(t) = \int_{-\infty}^t x(\tau) d\tau + \text{Constant}$

This system has Memory. It needs infinite memory; to have all values of $x(\tau)$ from $\tau = -\infty$ to $\tau = t$ to be able to calculate the output $y(t)$ at time t .

$$(2) \quad y(t) = a x(t) + b x(t - T_1) - c x(t - T_2)$$

This system have Finite Memory. That is to find value of $y(t)$ at time t it needs present value of input at times t , $t - T_1$, and $t - T_2$.

4. Causal Systems

A system is Causal if the output $y(t)$ at time t_0 depends only on current and previous values of the input, i.e. on $x(t)$ for $t \leq t_0$.

Non-Causal system is the one that its output also depends on future values of input.

Note all physical system are causal.

Note all the examples in previous page are Causal.

Ex * $y(t) = x(t^3)$ is Non-Causal

* $y(t) = x(5t)$ " " .

* $y(t) = a x^3(t)$ is Causal, but

* $y(t) = b x(t + T_1)$ is Non-Causal.
 non-linear.

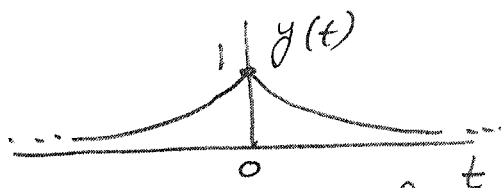
4.1 Causal Signals

Given a signal $h(t)$, then

- (1) $h(t)$ is causal if $h(t) = 0$ for $t < 0$
- (2) $h(t)$ is Anti-causal if $h(t) = 0$ for $t \geq 0$
- (3) $h(t)$ is Non-causal if $h(t)$ has non-zero values for $t \geq 0$ and $t < 0$.

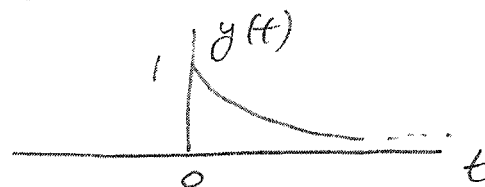
Ex

(a) $y(t) = e^{-|t|}$



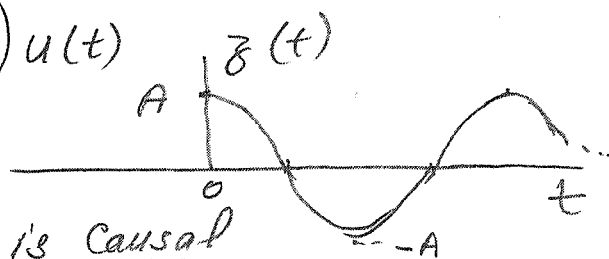
is Non-causal

(b) $y(t) = e^{-t} u(t)$



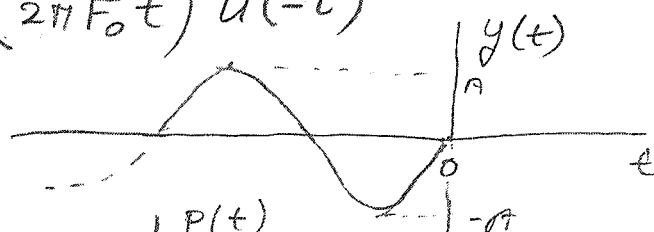
is Causal

$z(t) = A \cos(2\pi F_0 t) u(t)$

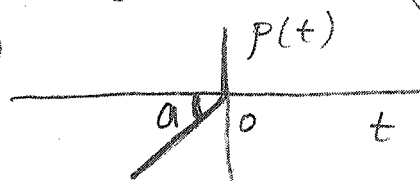


is Causal

(c) $y(t) = A \sin(2\pi F_0 t) u(-t)$



$p(t) = at u(-t)$



Anti-causal

5. Stability, Stable Systems

A signal $x(t)$ is bounded if $|x(t)| < B < \infty$ for all t .

Now

$$x(t) \rightarrow \boxed{\mathcal{T}} \rightarrow y(t) = \mathcal{T}\{x(t)\}$$

Bounded Input Bounded Output Stability (BIBO):

* System $\mathcal{T}$ is BIBO stable if and only if for any and all bounded input $x(t)$, the output is bounded.

i.e

For $|x(t)| < B < \infty$ then $|y(t)| < M < \infty$ for all t .

Ex

$$y(t) = a x(t), \quad y(t) = 5 x(t - T_1) - x(t + T_2)$$

$$y(t) = t x(t) \text{ for } t_1 \leq t \leq t_2 < \infty$$

$$y(t) = \int_0^{\infty} e^{-(t-\beta)} x(\beta) d\beta$$

Linear Time-Invariant (LTI) Systems

* In this course we only consider the Linear Time-Invariant (LTI) systems.

$$* \quad x(t) \rightarrow \boxed{\mathcal{T}} \rightarrow y(t) = \mathcal{T}\{x(t)\}$$

$$x(t) \xrightarrow{\mathcal{T}} y(t)$$

- * All the signals and system we discuss in this course are deterministic.
- * Input and output relation of a LTI system can be evaluated in several ways.
- * Each method characterizes the LTI continuous-time system uniquely.

Time-Domain

(a) Convolution Integral

(b) Differential Equation

(c) state variable

Transform-Domain

(a) Laplace domain

(b) Frequency domain

(c) state variable

* The main goal of analysis of a LTI system is to find a unique output of the system given a specific input.

- * Analysis and design of the system can be easier in one domain than in the other domain.
- * Initial Conditions of a system are its output values before we apply an input, $x(t)$.
- * The system $\mathcal{T}$ is called Relaxed or at Rest if its initial conditions are all zero when we apply the input $x(t)$.



Thus the output $y(t)$ is produced due to only applying the input $x(t)$ and nothing else.

- * A signal, $g(t)$, is called Absolutely Integrable over interval $[a, b]$ if

$$\int_{t=a}^b |g(t)| dt < \infty \quad , \text{ i.e. the integral exists and has finite value.}$$

Note that we can have $a = -\infty$, $b = \infty$, that is

$$\int_{-\infty}^{\infty} |g(t)| dt < \infty \quad \text{and } g(t) \text{ is}$$

absolutely integrable over all time.

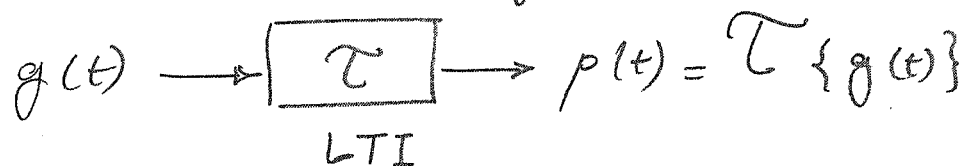
The Convolution Integral

* We know that for any signal, $x(t)$, we have

$$\int_{\alpha=-\infty}^{\alpha=\infty} x(\alpha) \delta(t-\alpha) d\alpha = x(t)$$

* Now let $x(t)$ be absolutely integrable.

* Consider the LTI system $\mathcal{T}$ as follows.

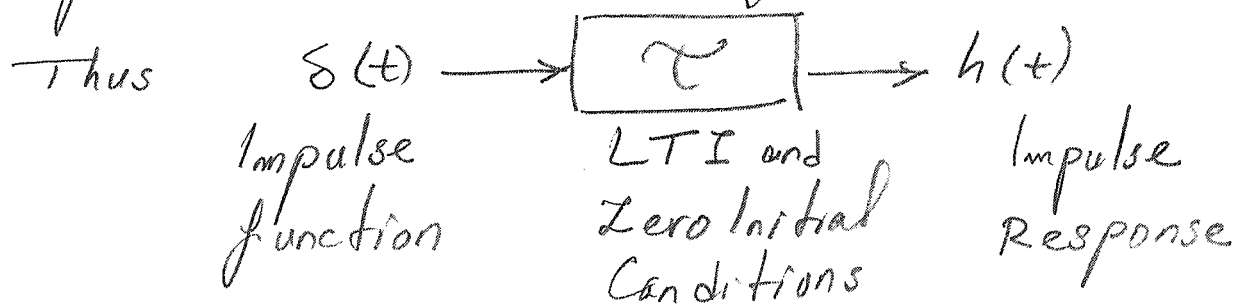


Definition of Impulse Response of System $\mathcal{T}$:

(1) If $\mathcal{T}$ is at rest, i.e. Zero Initial Conditions and

(2) the input to $\mathcal{T}$ is $\delta(t)$

The the output of $\mathcal{T}$ is called its Impulse Response and is denoted by $h(t)$.



$$\delta(t) \xrightarrow[\text{LTI, Relaxed}]{\mathcal{T}} h(t)$$

* Since the system is time-invariant, then we have

$$\delta(t-\alpha) \xrightarrow[\text{TI}]{\mathcal{T}} y(t, \alpha) = h(t-\alpha)$$

* Since $\mathcal{T}$ is Linear (i.e. Superposition properties hold), then

$$r(t, \alpha) = x(\alpha) \delta(t-\alpha) \xrightarrow[\text{LTI}]{\mathcal{T}} z(t, \alpha) = x(\alpha) h(t-\alpha)$$

Now if we add $r(t, \alpha)$ for all values of α we get

$$x(t) = \int_{\alpha=-\infty}^{\infty} x(\alpha) \delta(t-\alpha) \xrightarrow[\text{LTI}]{\mathcal{T}} y(t) = \int_{\alpha=-\infty}^{\infty} x(\alpha) h(t-\alpha) d\alpha$$

That is

$$x(t) \rightarrow \boxed{\mathcal{T}} \xrightarrow{\text{LTI Relaxed}} y(t) = \int_{\alpha=-\infty}^{\infty} x(\alpha) h(t-\alpha) d\alpha$$

$$x(t) \rightarrow \boxed{h(t)} \xrightarrow{\text{LTI Relaxed System}} y(t) = x(t) * h(t)$$

* $\int_{-\infty}^{\infty} x(\alpha) h(t-\alpha) d\alpha$ is the Linear

Convolution Integral of signals $x(t)$ and $h(t)$. It is also denoted by $x(t) * h(t)$.

* The Impulse Response $h(t)$ uniquely defines the LTI system $\mathcal{T}$.

* For any input $x(t)$ to the LTI Relaxed system $\mathcal{T}$, the output is $y(t) = x(t) * h(t)$.

Properties of the Linear Convolution Integral:

(1) Commutativity:

$$x(t) * h(t) = h(t) * x(t)$$

That is

$$\int_{-\infty}^{\infty} x(\alpha) h(t-\alpha) d\alpha = \int_{-\infty}^{\infty} h(\beta) x(t-\beta) d\beta$$

$$\text{for } t - \alpha = \beta$$

(2) Associativity:

$$\begin{aligned} x(t) * h_1(t) * h_2(t) &= [x(t) * h_1(t)] * h_2(t) \\ &= x(t) * [h_1(t) * h_2(t)] \end{aligned}$$

(3) Distributivity:

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

That is

$$\begin{array}{c} x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \\ = \\ h(t) \rightarrow \boxed{x(t)} \rightarrow y(t) \end{array}$$

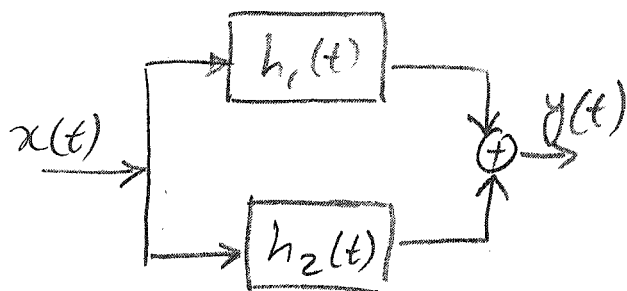
Commutativity

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t) = x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$$

Cascaded
systems

Associativity

$$h_e(t) = h_1(t) * h_2(t)$$



Parallel systems

$$= x(t) \rightarrow \boxed{h_1(t) + h_2(t)} \rightarrow y(t)$$

Distributivity $h_e(t) = h_1(t) + h_2(t)$

$$(4) \quad y(t) = x(t) * h(t) \quad , \quad y(t-a) = x(t-a) * h(t)$$

$$y(t-b) = x(t) * h(t-b),$$

$$\text{and } y(t-b-a) = x(t-a) * h(t-b)$$

$$(5) \quad \frac{d}{dt} y(t) = \left[\frac{dx(t)}{dt} \right] * h(t) = x(t) * \left[\frac{dh(t)}{dt} \right]$$

$$(6) \quad x(at) * h(at) = \frac{1}{|a|} y(at)$$

Ex 1 $h(t) = \delta(t)$, then $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\alpha) \delta(t-\alpha) d\alpha$
 $= x(t)$

Ex 2 $h(t) = \delta(t-\beta)$ then

$$x(t) * \delta(t-\beta) = \int_{-\infty}^{\infty} x(\alpha) \delta(t-\alpha-\beta) d\alpha = x(t-\beta)$$

Since $\delta(t-\alpha-\beta) = 1$ only when $t-\alpha-\beta = 0$
 or $\alpha = t-\beta$, $\delta(t-\alpha-\beta) = 0$ if $t-\alpha-\beta \neq 0$.

Ex 3 $h(t) = u(t)$ then

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\alpha) u(t-\alpha) d\alpha$$

but $u(t-\alpha) = 1$ only when $t-\alpha \geq 0$ or $t \geq \alpha$
 otherwise $u(t-\alpha) = 0$ for $t-\alpha < 0$ or $t < \alpha$.

Thus
$$\int_{-\infty}^{\infty} x(\alpha) u(t-\alpha) d\alpha = \int_{-\infty}^t x(\alpha) d\alpha$$

 i.e. Integrator

Ex 4 Let $h(t) = e^{-\beta t}$ for $t \geq 0$ and $\beta > 0$.
Also let $x(t) = u(t)$.

$h(t)$ is causal signal and we can write it as

$$h(t) = e^{-\beta t} u(t), \text{ for } \beta > 0.$$

Now the output $y(t) = x(t) * h(t)$ is

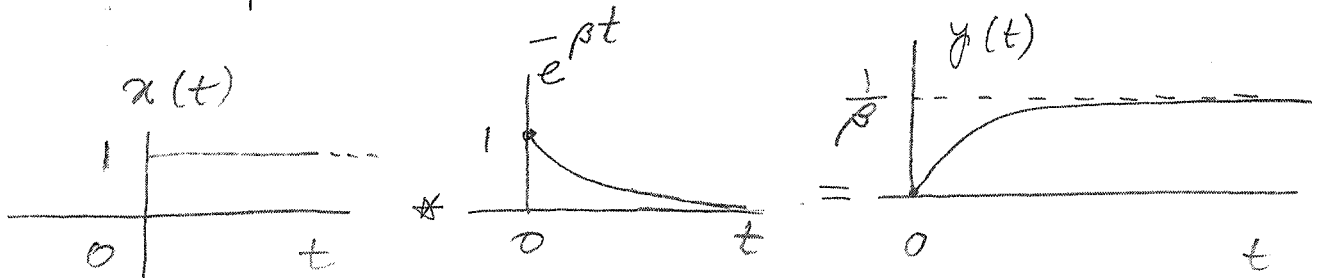
$$y(t) = \int_{\alpha=-\infty}^{\infty} h(\alpha) x(t-\alpha) d\alpha = \int_{\alpha=-\infty}^{\infty} e^{-\beta\alpha} \underbrace{u(\alpha)}_{\alpha \geq 0} \underbrace{u(t-\alpha)}_{\substack{t-\alpha \geq 0 \\ t \geq \alpha}} d\alpha$$

Thus $0 \leq \alpha \leq t$

Therefore

$$y(t) = \int_{\alpha=0}^t e^{-\beta\alpha} d\alpha = -\frac{1}{\beta} e^{-\beta\alpha} \Big|_{\alpha=0}^t =$$

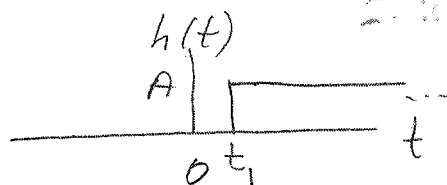
$$y(t) = -\frac{1}{\beta} \{ e^{-\beta t} - 1 \} = \frac{1}{\beta} (1 - e^{-\beta t}) \text{ for } t \geq 0$$



Ex 5

$$h(t) = A u(t - t_1)$$

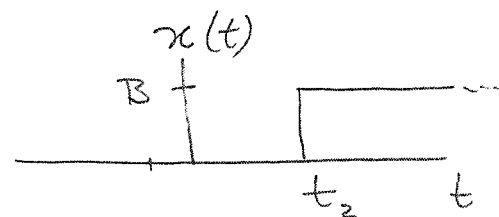
$$t_1 > 0$$



and

$$x(t) = B u(t - t_2)$$

$$t_2 > t_1 > 0$$



Find $y(t) = x(t) * h(t)$ for $t \geq 0$.

$$y(t) = \int_{-\infty}^{\infty} h(\alpha) x(t - \alpha) d\alpha = y(t) u(t)$$

i.e. for $t \geq 0$

$$y(t) u(t) = \int_{-\infty}^{\infty} A u(\alpha - t_1) B u(t - \alpha - t_2) d\alpha$$

Let's make change of variable

$$\alpha - t_1 = \beta \quad d\alpha = d\beta \quad \text{and for}$$

$$-\infty \leq \alpha \leq \infty \Rightarrow -\infty \leq \beta \leq \infty. \text{ Thus}$$

$$y(t) u(t) = \int_{\beta=-\infty}^{\infty} AB \underbrace{u(\beta)}_{\beta \geq 0} u(t - t_1 - t_2 - \beta) d\beta$$

$$0 \leq \beta \leq t - t_1 - t_2$$

$$0 \leq t - t_1 - t_2$$

$$y(t) u(t - t_1 - t_2) = \int_0^{t - t_1 - t_2} AB d\beta = AB (t - t_1 - t_2)$$

Ex 6 $h(t) = A e^{-at}$, $t \geq 0$, $a > 0$ and
 $x(t) = B e^{-bt}$, $t \geq 0$, $b > 0$.

Find $y(t) = x(t) * h(t)$.

$$y(t) = \int_{\beta=-\infty}^{\infty} h(\beta) x(t-\beta) d\beta =$$

$$y(t) = \int_{\beta=-\infty}^{\infty} A e^{-a\beta} \underbrace{u(\beta)}_{\beta \geq 0} B e^{-b(t-\beta)} \underbrace{u(t-\beta)}_{\substack{t-\beta \geq 0 \\ t \geq \beta \geq 0 \\ t \geq 0}} d\beta =$$

$$y(t) = \int_{\beta=0}^t AB e^{-a\beta} e^{-b(t-\beta)} d\beta =$$

$$y(t) = AB e^{-bt} \int_{\beta=0}^t e^{(b-a)\beta} d\beta$$

$$= AB e^{-bt} \left\{ \left(\frac{1}{b-a} \right) e^{(b-a)\beta} \right\}_{\beta=0}^t$$

$$y(t) = \left(\frac{AB}{b-a} \right) e^{-bt} \left\{ e^{(b-a)t} - 1 \right\}$$

$$y(t) = \left(\frac{AB}{b-a} \right) \left\{ e^{-at} - e^{-bt} \right\} \text{ for } t \geq 0$$

Hints for Convolution Integral: $y(t) = x(t) * h(t)$

- (1) specify the range of non-zero values of $x(t)$ and $h(t)$, hence the range of convolution integration. You can use elementary signals such as $u(t)$ - the step function.
- (2) You can break down the integral to sum of several integrals, since
- (3) Be aware of and use properties of LTI, and Convolution Integral which is a linear Time Invariant operation.

Graphical Interpretation of Convolution operation

Given $x(t)$, $h(t)$, Find $y(t) = x(t) * h(t)$
for $-\infty < t < \infty$.

step 1

Let $t = \alpha$ and denote $x(\alpha)$
and $h(\alpha)$

step 2 Fold (Reflect) either
one of the $x(\alpha)$ or $h(\alpha)$
to obtain for example
 $h(\alpha)$ and $x(-\alpha)$.

step 3 shift the folded signal
by t . e.g. to obtain
 $x(t - \alpha)$.

step 4 Determine the range of
signals, e.g. $h(\alpha)$ and
 $x(t - \alpha)$.

step 5 multiply $h(\alpha)$ and
 $x(t - \alpha)$

Step 6

over non-zero ranges of $h(\alpha)x(t-\alpha)$,
that is the overlap areas of $h(\alpha)x(t-\alpha)$,
compute the integral of the product
for all α as t varies. for its possible
values.

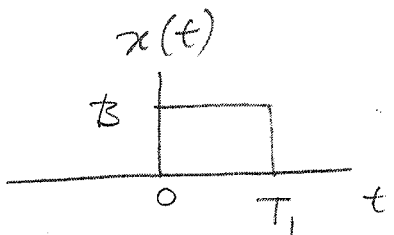
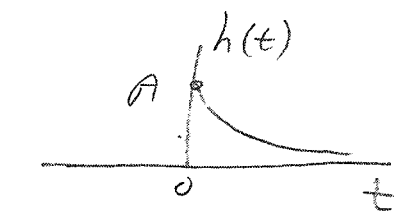
Ex 1

$$h(t) = A e^{-at} \quad \text{for } t \geq 0$$

$$\text{or } h(t) = A e^{-at} u(t)$$

and

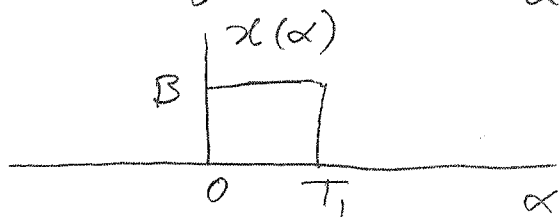
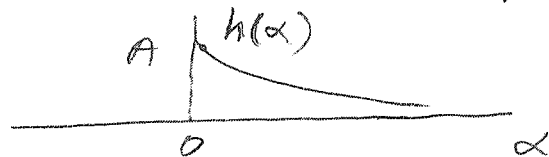
$$x(t) = \begin{cases} B & 0 \leq t \leq T_1 \\ 0 & \text{otherwise} \end{cases}$$



Find $y(t) = x(t) * h(t)$ for all
 $-\infty \leq t \leq \infty$.

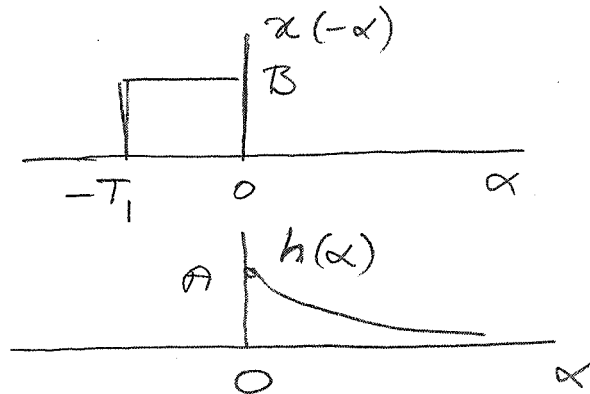
* We can write $x(t) = B u(t) - B u(t - T_1)$

Step 1 $h(\alpha) = A e^{-a\alpha} u(\alpha)$, $x(\alpha) = B u(\alpha) - B u(\alpha - T_1)$

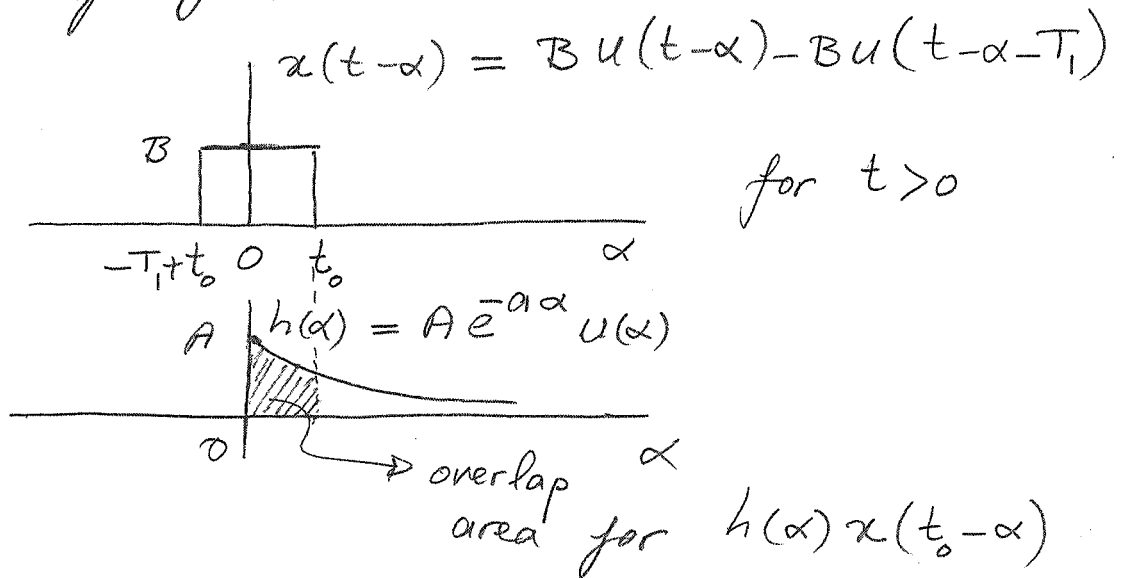


$$T_1 > 0$$

Step 2 Fold $x(\alpha)$ to obtain $x(-\alpha)$:



Step 3 shift by t ,



Step 4

$$y(t_0) = \int_{\alpha=-\infty}^{\infty} h(\alpha) x(t_0-\alpha) d\alpha$$

$$y(t_0) = \int_{\alpha=-\infty}^{\infty} A e^{-a\alpha} u(\alpha) [B u(t_0-\alpha) - B u(t_0-\alpha-T_1)] d\alpha$$

$$y(t_0) = AB \left\{ \int_{\alpha=-\infty}^{\infty} e^{-a\alpha} u(\alpha) u(t_0-\alpha) d\alpha - \int_{\alpha=-\infty}^{\infty} e^{-a\alpha} u(\alpha) u(t_0-\alpha-T_1) d\alpha \right\}$$

Thus we have Two Convolution integrals such that

$$y(t_0) = y_1(t_0) - y_2(t_0)$$

step 5

$$y_1(t_0) = AB \int_{\alpha=-\infty}^{\infty} e^{-a\alpha} \underbrace{u(\alpha)}_{\alpha \geq 0} \underbrace{u(t_0-\alpha)}_{t_0-\alpha \geq 0} d\alpha$$

$t_0 \geq \alpha \geq 0, t_0 \geq 0$

$$y_2(t_0) = AB \int_{\alpha=-\infty}^{\infty} e^{-a\alpha} \underbrace{u(\alpha)}_{\alpha \geq 0} \underbrace{u(t_0-\alpha-T_1)}_{t_0-\alpha-T_1 \geq 0} d\alpha$$

$t_0 - T_1 \geq \alpha \geq 0$

$$t_0 - T_1 \geq 0, t_0 \geq T_1$$

Now for $t_0 \geq 0$:

$$y_1(t_0) = AB \int_{\alpha=0}^{t_0} e^{-a\alpha} d\alpha = AB \left(\frac{-1}{a} \right) \left[e^{-a\alpha} \right]_{\alpha=0}^{t_0}$$

$$y_1(t_0) = AB \left(\frac{1}{a} \right) \left[1 - e^{-at_0} \right], t_0 \geq 0$$

And for $y_2(t_0)$ we have for $t_0 \geq T_1$:

$$y_2(t_0) u(t_0 - T_1) = AB \int_{\alpha=0}^{t_0 - T_1} e^{-a\alpha} d\alpha = AB \left(\frac{-1}{a} \right) \left[e^{-a\alpha} \right]_{\alpha=0}^{\alpha=t_0 - T_1}$$

$$y_2(t) u(t_0 - T_1) = AB \left(\frac{1}{a} \right) \left[1 - e^{-a(t_0 - T_1)} \right], t_0 \geq T_1 \geq 0$$

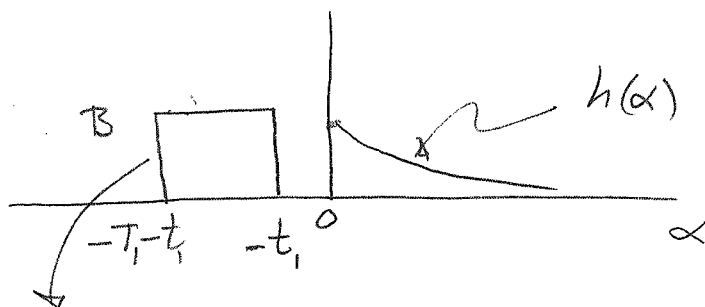
Thus, for $t = t_0 \geq 0$

$$y(t) = y_1(t) \cdot u(t) - y_2(t) u(t - T_1)$$

$$y(t) = AB \left(\frac{1}{a} \right) \left\{ u(t) - e^{-at} u(t) - u(t - T_1) + e^{-a(t - T_1)} u(t - T_1) \right\}$$

for $t \geq 0$.

Note that for $t < 0$



$$x(-t_1 - \alpha) = B u(-t_1 - \alpha) - B u(-t_1 - \alpha - T_1)$$

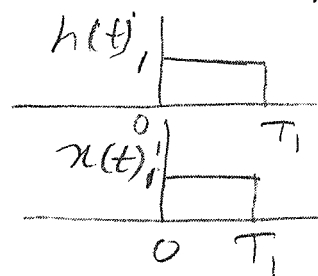
There is no overlap between $h(\alpha)$ and $x(-t_1 - \alpha)$, resulting in

$$y(t) = x(t) * h(t) = 0 \text{ for } t < 0.$$

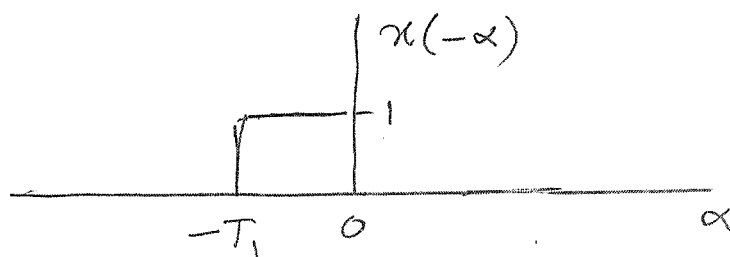
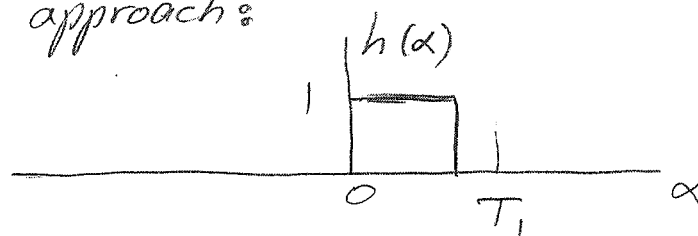
Ex 2

$$h(t) = u(t) - u(t - T_1)$$

$$x(t) = u(t) - u(t - T_1)$$



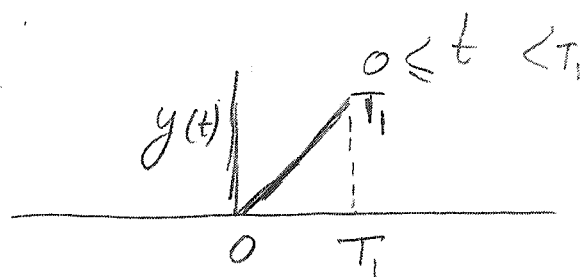
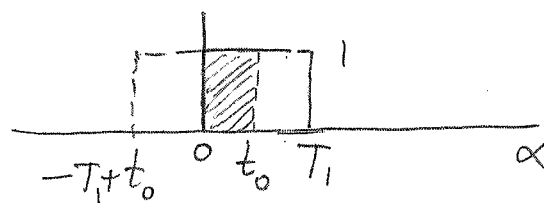
Graphical approach:

 $t < 0$:

For any shift of $x(-\alpha)$ to the left, i.e. $x(-t-\alpha)$ there is no overlap.

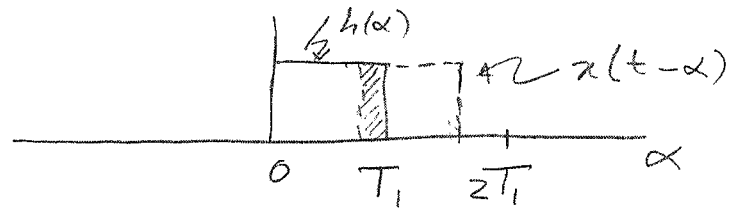
Thus

$$y(t) = x(t) * h(t) = 0$$

for $t < 0$. $t \geq 0$
 $y(0) = 0$ no overlap
For $T_1 > t = t_0 > 0$:

$$y(t) = t \quad \text{for } 0 \leq t \leq T_1$$

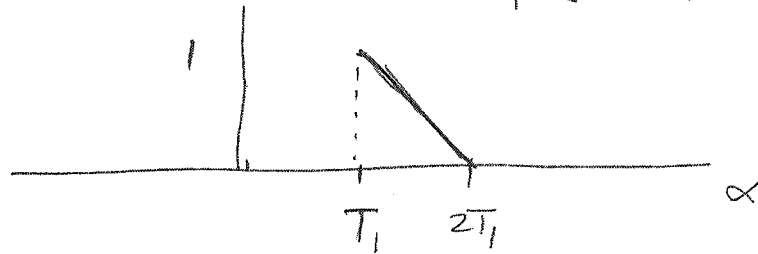
For $T_1 \leq t_0 \leq 2T_1$:



Thus

$$y(t) = 2T_1 - t$$

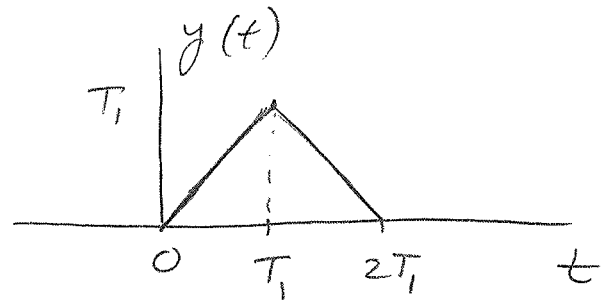
$$T_1 \leq t \leq 2T_1$$



And for $t > 2T_1 \Rightarrow y(t) = 0$

Thus

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \leq t \leq T_1 \\ 2T_1 - t & \text{for } T_1 \leq t \leq 2T_1 \\ 0 & \text{for } t \geq 2T_1 \end{cases}$$



Some Properties of LTI systems

P-1) Causal LTI systems:

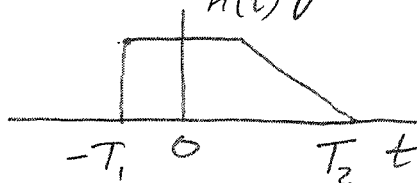
$$x(t) \rightarrow \boxed{h(t)}_{\text{LTI}} \rightarrow y(t) = h(t) * x(t)$$

The LTI is causal if $h(t) = 0$ for $t < 0$.

$$\begin{aligned} \text{Then } y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) d\alpha \\ &= \int_{-\infty}^t x(\alpha) h(t-\alpha) d\alpha \end{aligned}$$

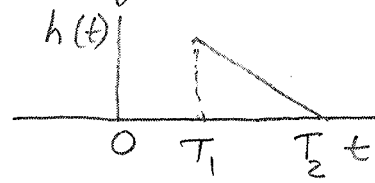
P-2) Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) systems:

* The system is FIR if $h(t)$ is of finite duration.



FIR

Non-Causal



FIR

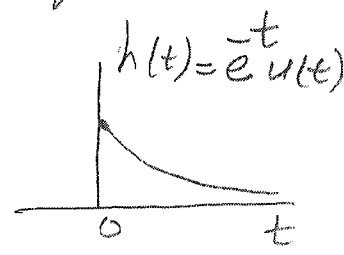
Causal

* The system is IIR if $h(t)$ is of infinite duration.



IIR

Non-Causal



IIR

Causal

P-3) The LTI system is BIBO stable if $h(t)$ is absolutely integrable (summable),
That is if
$$\int_{-\infty}^{\infty} |h(\alpha)| d\alpha \leq M_h < \infty$$

* Note LTI FIR system is always BIBO stable.

Ex $h(t) = A e^{-t} u(t)$ is BIBO stable because

$$\begin{aligned} \int_{-\infty}^{\infty} |h(\alpha)| d\alpha &= \int_0^{\infty} |A e^{-\alpha}| d\alpha = |A| \int_0^{\infty} |e|^{-\alpha} d\alpha \\ &= |A| \left(-\frac{1}{\alpha} \right) \left\{ |e|^{-\alpha} \right\}_{\alpha=0}^{\infty} = |A| \left(\frac{1}{\alpha} \right) < \infty. \end{aligned}$$

* $h(t) = B u(t + T_1)$ is not BIBO stable because

$$\begin{aligned} \int_{-\infty}^{\infty} |h(\alpha)| d\alpha &= \int_{-T_1}^{\infty} |B| d\alpha = |B| \int_{-T_1}^{\infty} d\alpha \\ &= |B| \left\{ \alpha \right\}_{-T_1}^{\infty} = |B| \left\{ \infty - T_1 \right\} = \infty. \end{aligned}$$

* $h(t) = \frac{-|t|}{e}$ $-\infty < t < \infty$ is BIBO stable.

* $h(t) = e^{-\alpha t} u(t)$ for $\alpha < \infty$ is NOT BIBO stable.

P-4) Invertible LTI systems.

The given LTI system $h(t)$ is Invertible if its inverse is BIBO stable.

$$\text{That is, } h(t) * g(t) = g(t) * h(t) = \delta(t)$$

$h(t)$ is invertible if $g(t)$ exists and is BIBO stable.

Ex

$$h(t) = \delta(t - T_1) \text{ the}$$

$$g(t) = \delta(t + T_1) \text{ is BIBO stable}$$

(but Non-causal).

$$\begin{aligned} h(t) * g(t) &= \int_{-\infty}^{\infty} h(\alpha) g(t - \alpha) d\alpha \\ &= \int_{-\infty}^{\infty} \delta(\alpha - T_1) \delta(t - \alpha + T_1) d\alpha \end{aligned}$$

For $\alpha - T_1 = \beta$:

$$= \int_{-\infty}^{\infty} \delta(\beta) \delta(t - \beta) d\beta = \delta(t)$$